Ph.D. defense
Perpignan, December 10th, 2014

Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

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CNRS, LIRMM, UMR 5506

Context

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Around 30% of an airplane's cost, around 40% of a car's cost



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How to implement these computational tasks in embedded systems?

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Fixed-point computations

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But, how to make it easy, fast, and numerically safe to use by non-expert programmers?

State of the art on how to achieve this objective?

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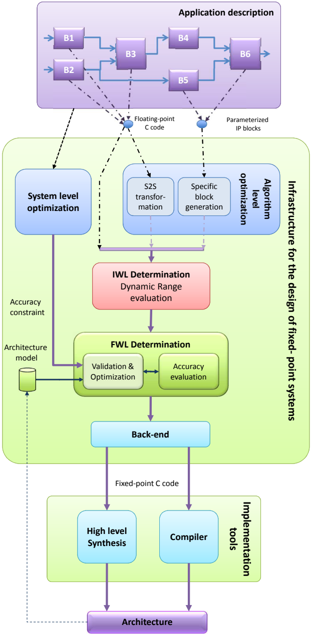

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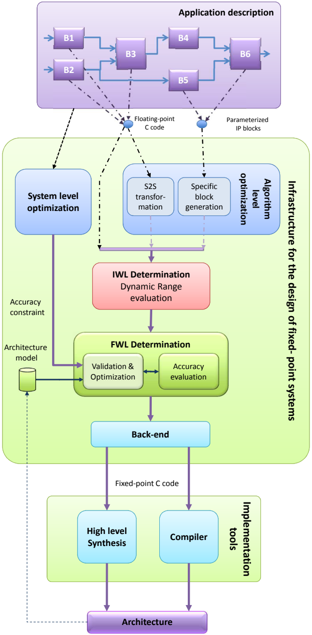

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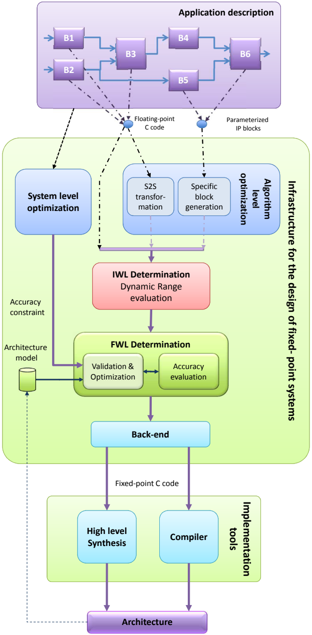
- **Long term objective:** code synthesis for matrix inversion



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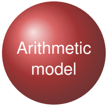

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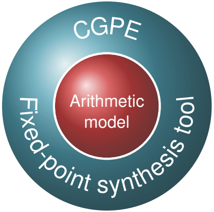

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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
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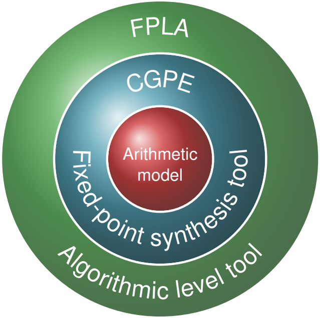
     - trade-off implementations for matrix multiplication
     - code synthesis for Cholesky decomposition and triangular matrix inversion



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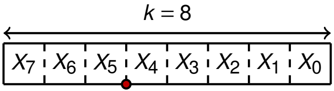
- ▷ X the k-bit integer representation of x
- ▷ f the implicit scaling factor of x



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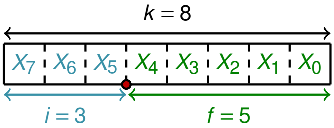

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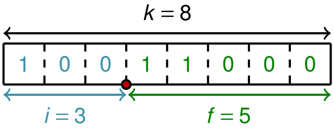

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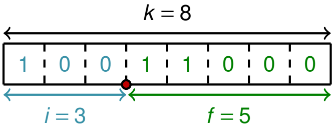

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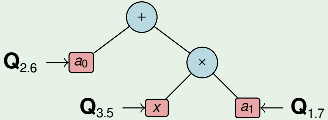

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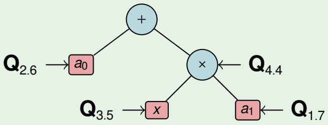

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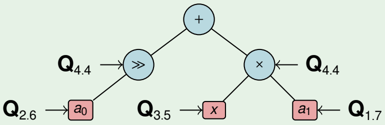

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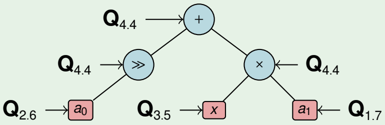

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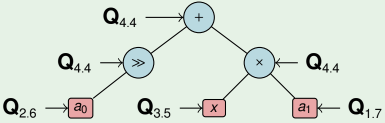

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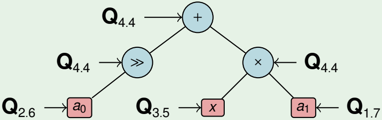

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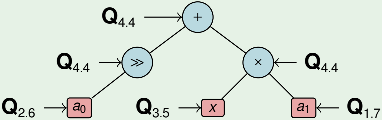

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Even for small problems, this process is tedious: How to automate it?

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- For each coefficient or variable v, we keep track of 2 intervals **Val**(v) and **Err**(v)
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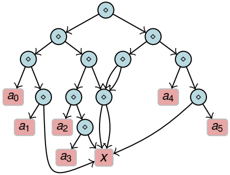

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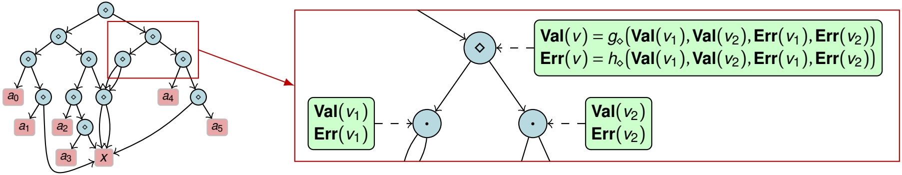

How to propagate $\mathbf{Val}(v)$ and $\mathbf{Err}(v)$ for $\diamond \in \{+, -, \times, \ll, \gg, \sqrt{\ }, /\}$?

Fixed-point addition

- The two variables v_1 and v_2 must be aligned: in the same fixed-point format $\mathbf{Q}_{i.f}$

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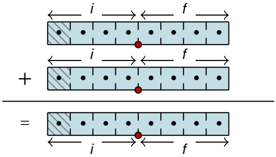

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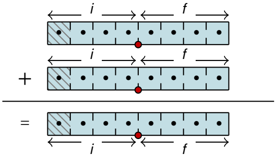

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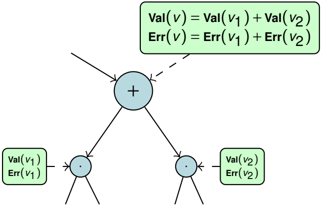

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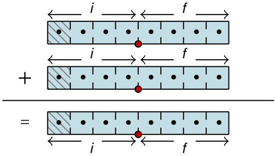

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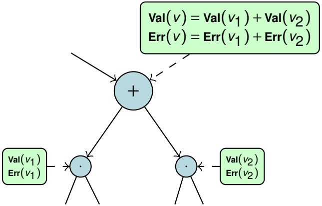

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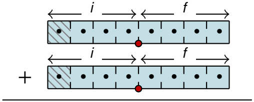

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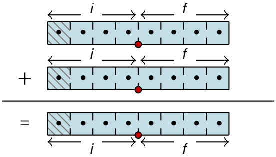

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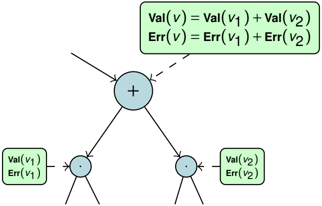

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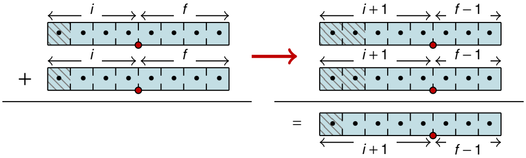

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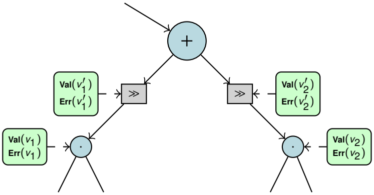

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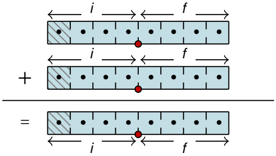

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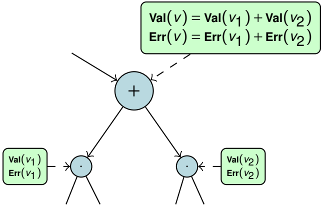

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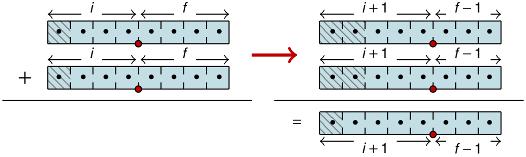

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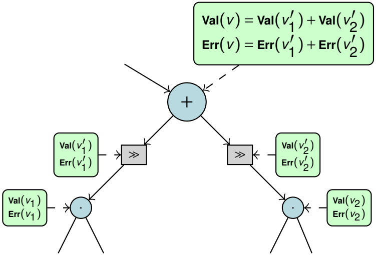

Fixed-point multiplication

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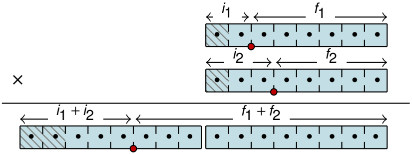

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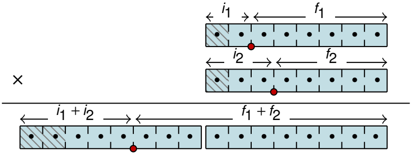

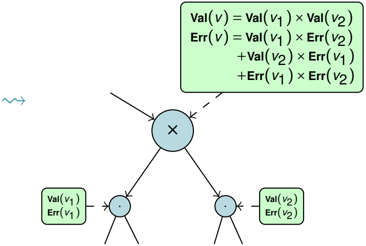

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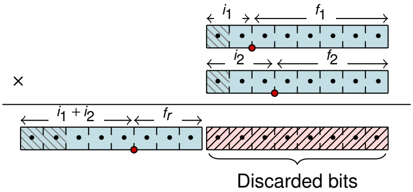

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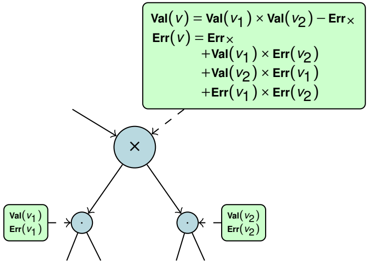

- $\mathbf{Err}_\times = \left[0, 2^{-f_r} - 2^{-(f_1+f_2)}\right]$

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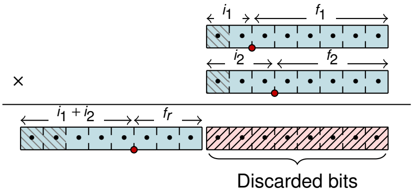
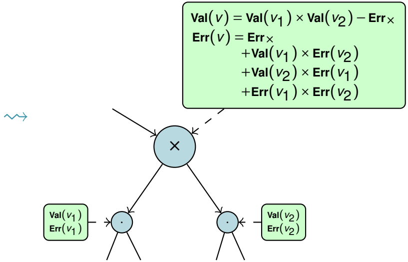


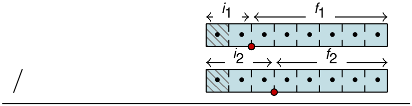


- $\mathbf{Err}_\times = \left[0, 2^{-f_r} - 2^{-(f_1+f_2)}\right]$
- This multiplication is available on integer processors and DSPs

```
int32_t mul (int32_t v1, int32_t v2){
  int64_t prod = ((int64_t) v1) * ((int64_t) v2);
  return (int32_t) (prod >> 32);
}
```

Our new fixed-point division

- The output integer part of $\mathbf{Q}_{i_1.f_1}/\mathbf{Q}_{i_2.f_2}$ may be as large as $i_1 + f_2$



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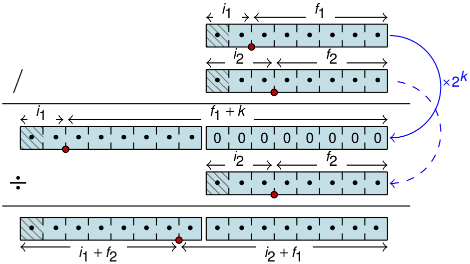

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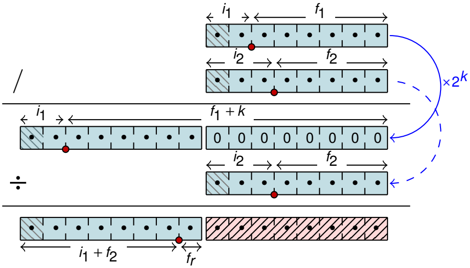

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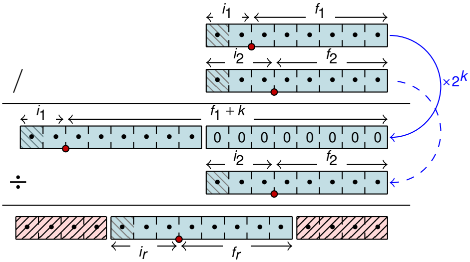
- How to obtain sharper a error bounds on $\mathbf{Err}_{/}$?



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  - ☹ risk of overflow at run-time

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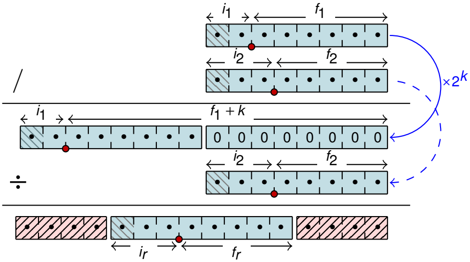

- $\mathbf{Err}_{/} = [-2^{f_r}, 2^{f_r}]$
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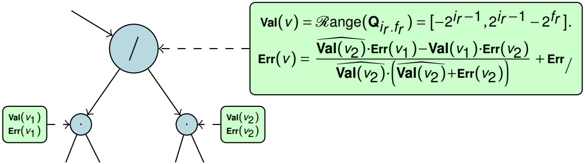
How to decide of the output format of division?

- A large integer part
  - ✓ prevents overflow
  - ✗ loose error bounds and loss of precision
- A small integer part
  - ✗ may cause overflow
  - ✓ sharp error bounds and more accurate computations

The propagation rule and implementation of division

- Once the output format decided $\mathbf{Q}_{i_r.f_r}$



- $\widetilde{\mathbf{Val}(v_2)} = \dfrac{\mathbf{Val}(v_1)}{\widetilde{\mathbf{Val}(v)} + \mathbf{Err}_/} \cap \mathbf{Val}(v_2)$ and $\widetilde{\mathbf{Val}(v)} = [-2^{i_r-1}, -2^{-f_r}] \cup [2^{-f_r}, 2^{i_r-1} - 2^{f_r}]$

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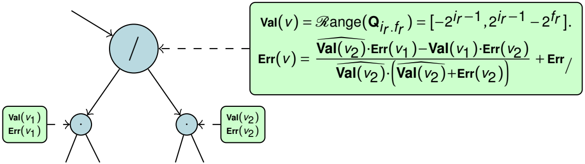

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```
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{
  int64_t t1 = ((int64_t)V1) << eta;
  int64_t V  = t1 / V2;


  return (int32_t) V;
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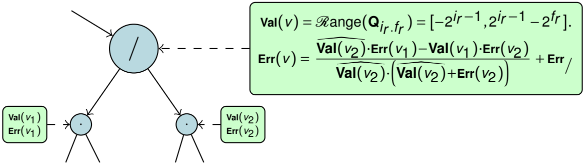

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  int64_t V  = t1 / V2;
  CGPE_ASSERT((((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
          || ((V & 0xFFFFFFFF80000000ll) == 0)));
  return (int32_t) V;
}
```

- Additional code to check for run-time overflows

The division format trade-off: case of inverting 2×2 matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbf{Q}_{2.30}$

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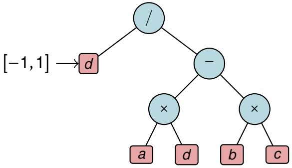

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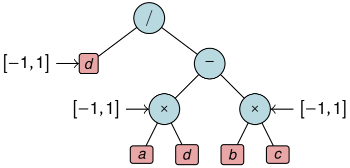

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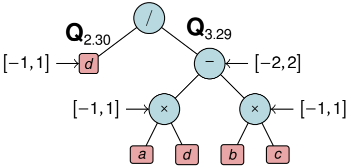

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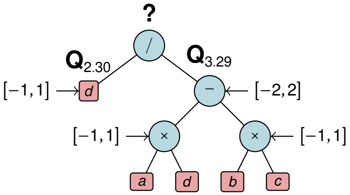

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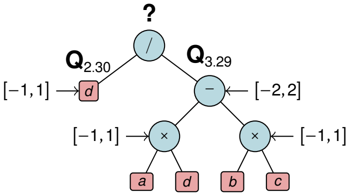

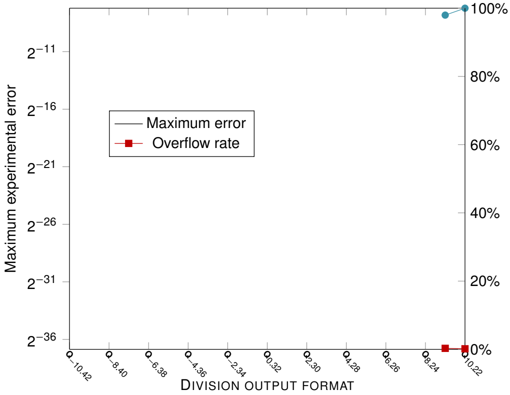

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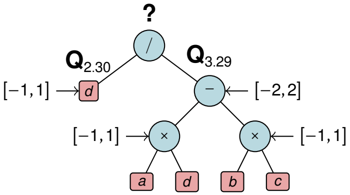

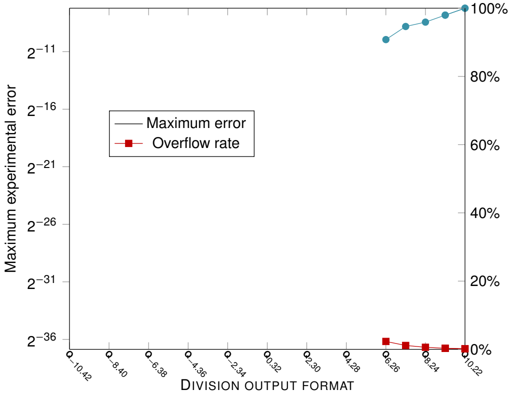

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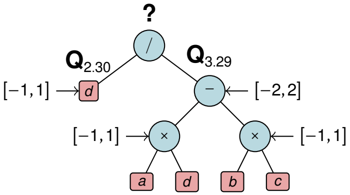

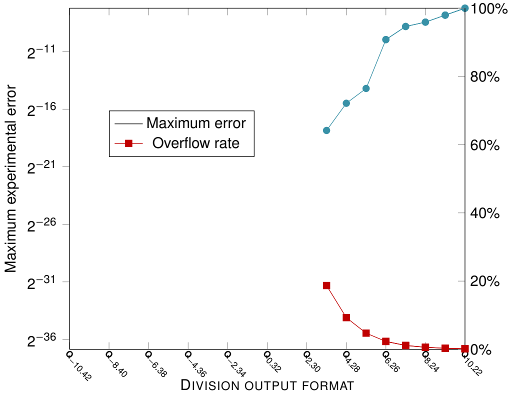

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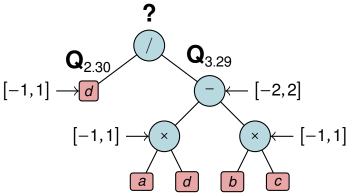

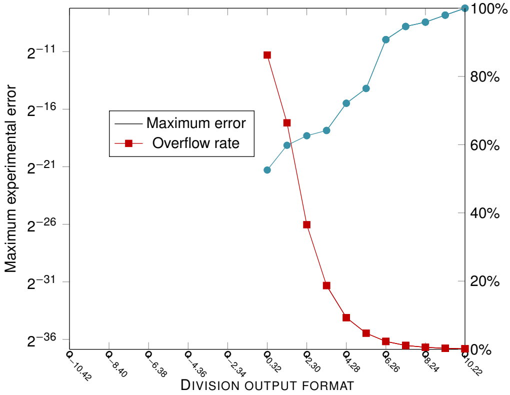

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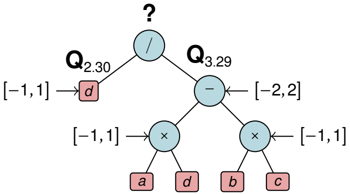

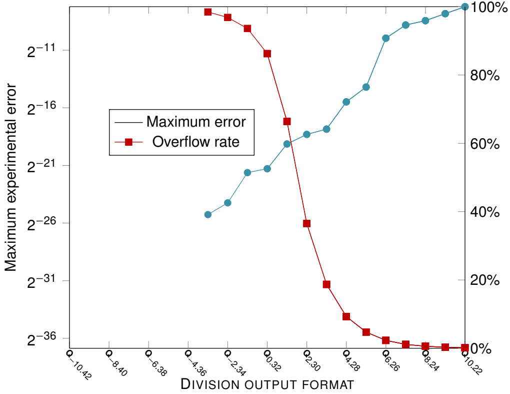

The division format trade-off: case of inverting 2×2 matrices

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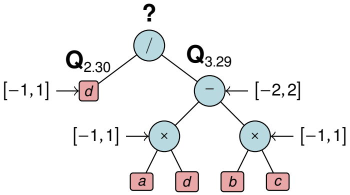

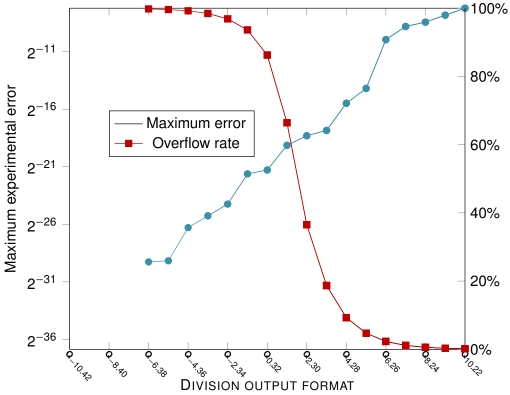

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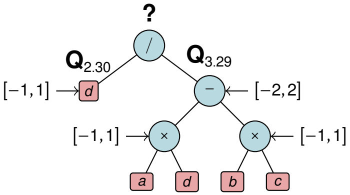

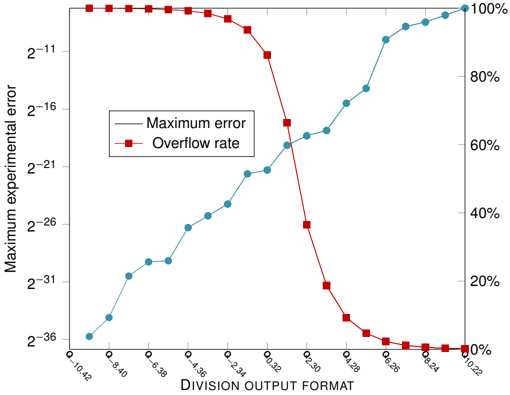

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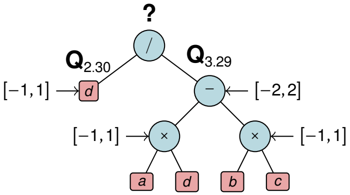

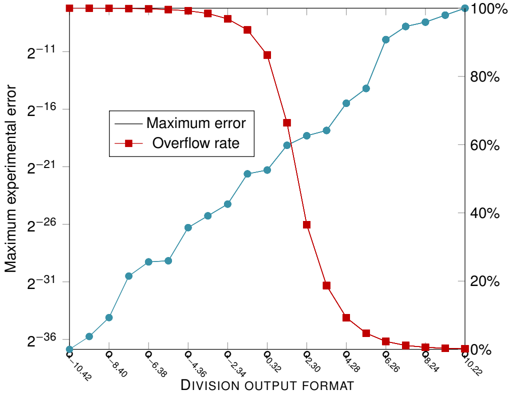

Outline of the talk

The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [?]
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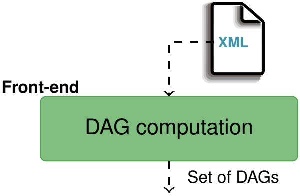

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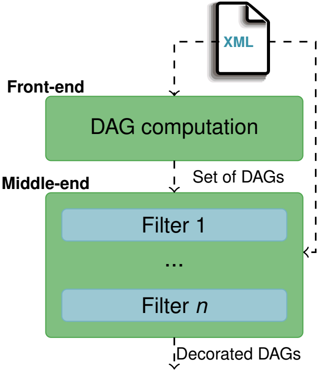

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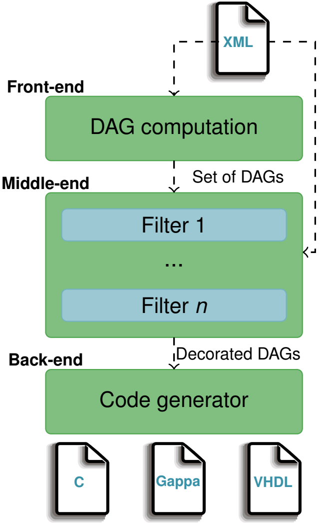
   - generates C codes and Gappa accuracy certificates



Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

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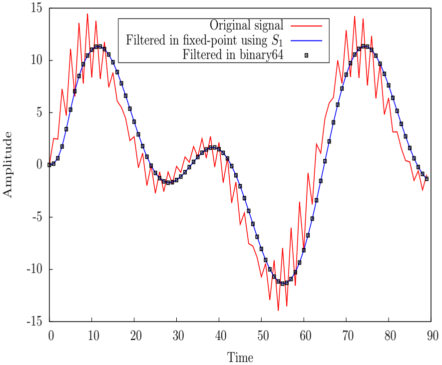

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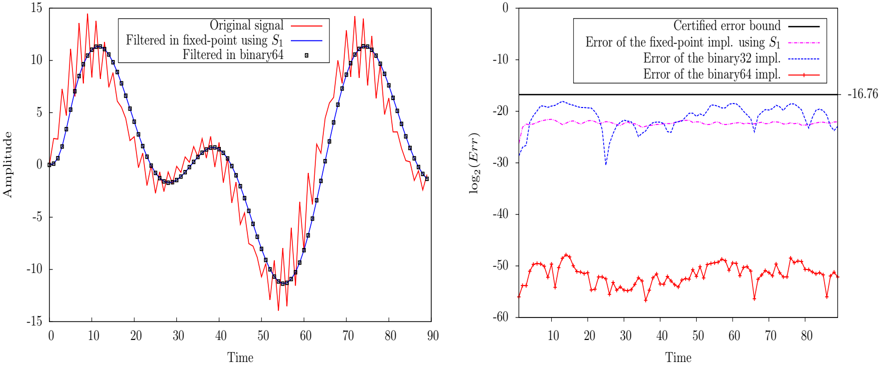

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```
int32_t filter( int32_t u0    /*Q5.27*/ , int32_t u1    /*Q5.27*/ ,
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                int32_t y1    /*Q6.26*/ , int32_t y2    /*Q6.26*/ ,
                int32_t y3    /*Q6.26*/ )
{                                               //Formats  Err
  int32_t r0  = mul(0x4a5cdb26, y1);            //Q8.24   [-2^{-24},0]
  int32_t r1  = mul(0xa6eb5908, y2);            //Q7.25   [-2^{-25},0]
  int32_t r2  = mul(0x4688a637, y3);            //Q5.27   [-2^{-27},0]
  int32_t r3  = mul(0x65718e3b, u0);            //Q2.30   [-2^{-30},0]
  int32_t r4  = mul(0x65718e3b, u3);            //Q2.30   [-2^{-30},0]
  int32_t r5  = r3 + r4;                        //Q2.30   [-2^{-29},0]
  int32_t r6  = r5 >> 2;                        //Q4.28   [-2^{-27.6781},0]
  int32_t r7  = mul(0x4c152aad, u1);            //Q4.28   [-2^{-28},0]
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  int32_t r9  = r7 + r8;                        //Q4.28   [-2^{-27},0]
  int32_t r10 = r6 + r9;                        //Q4.28   [-2^{-26.2996},0]
  int32_t r11 = r10 >> 1;                       //Q5.27   [-2^{-25.9125},0]
  int32_t r12 = r2 + r11;                       //Q5.27   [-2^{-25.3561},0]
  int32_t r13 = r12 >> 2;                       //Q7.25   [-2^{-24.3853},0]
  int32_t r14 = r1 + r13;                       //Q7.25   [-2^{-23.6601},0]
  int32_t r15 = r14 >> 1;                       //Q8.24   [-2^{-23.1798},0]
  int32_t r16 = r0 + r15;                       //Q8.24   [-2^{-22.5324},0]
  int32_t r17 = r16 << 2;                       //Q6.26   [-2^{-22.5324},0]
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Instruction selection to handle advanced instructions

- It is a well known problem in compilation $\rightsquigarrow$ proven to be NP-complete on DAGs
- Usually solved using a tiling algorithm:
  - input:
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Implementation in CGPE

- XML architecture description file that contains for each instruction:
  - its name, its type (signed or unsigned), its latency (# cycles),
  - a description of the pattern it matches,
  - a C macro to emulate it in software,
  - and a piece of Gappa script to compute the error entailed by its evaluation in fixed-point arithmetic.

The NOLTIS-like tiling algorithm

Near-Optimal Instruction Selection algorithm [?]

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

- Example: how to evaluate $a_0 + \Big((a_1 \cdot x) + \big((a_2 \cdot (x \cdot x)) \ll 1\big)\Big)$?

addition / shift $\rightsquigarrow$ 1 cycle

shift-and-add $\rightsquigarrow$ 1 cycle

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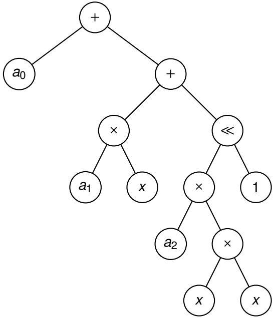

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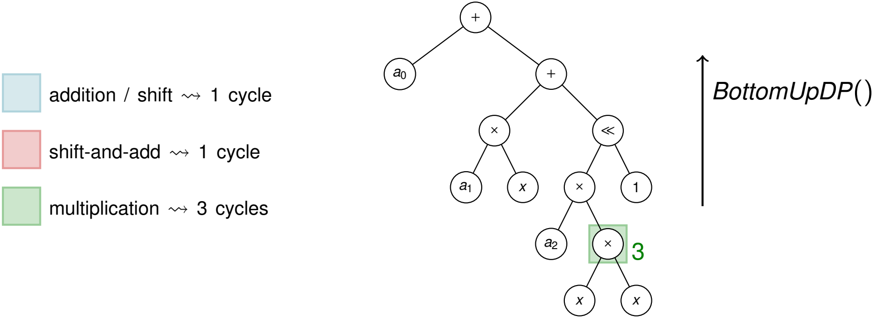

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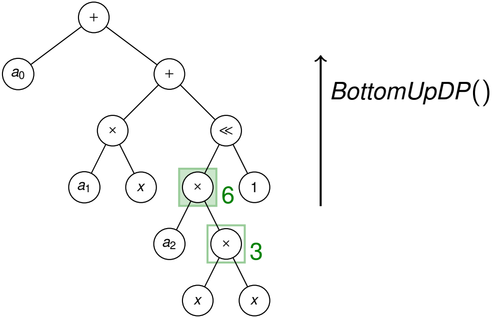

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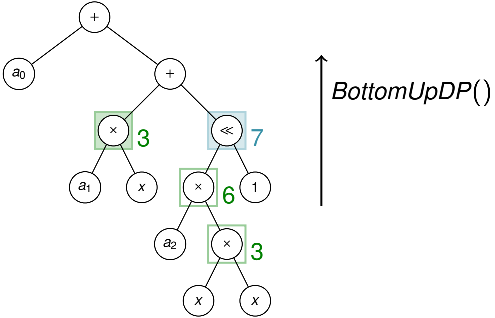

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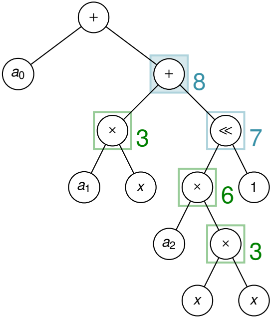

BottomUpDP()

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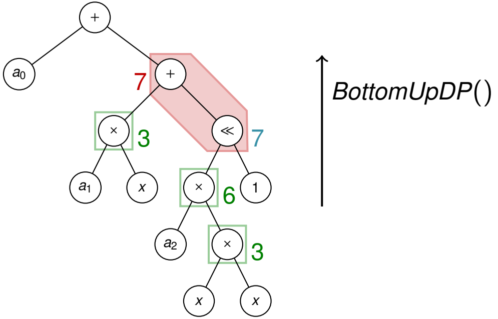

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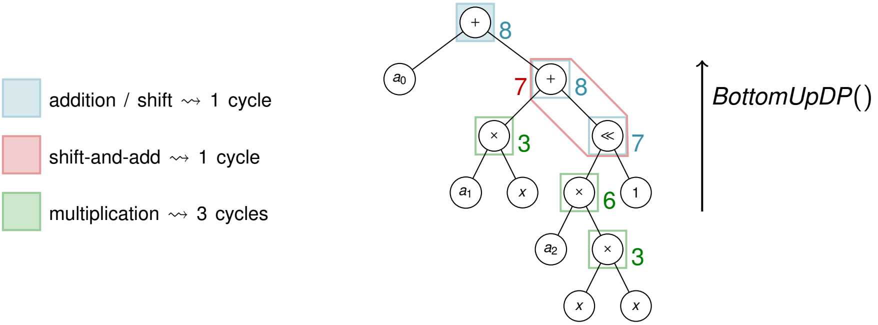

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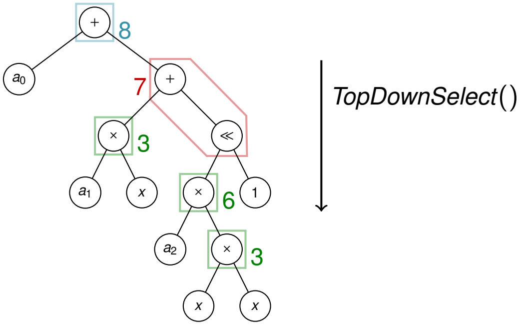

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- In our case, only the first step of NOLTIS is valuable.

- NOLTIS algorithm mainly relies on the evaluation of a cost function. We have implemented three different cost functions:
  - number of operator (regardless common subexpressions)
  - evaluation latency on unbounded parallelism
  - evaluation accuracy, computed using the Gappa script of each instruction

Impact on the number of instructions

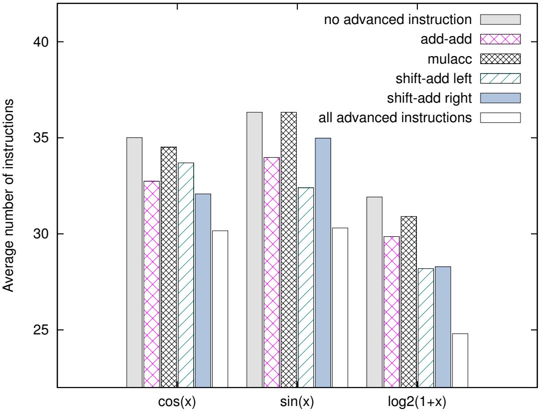

Figure: Average number of instructions in 50 synthesized codes, for the evaluation of polynomials of degree 5 up to 12 for various elementary functions.

- Remark 1: average reduction of 8.7 % up to 13.75 %
- Remark 2: interest of ST231 shift-and-add with left shift for $\sin(x)$ implementation
  $\rightsquigarrow$ reduction of 8.7 %
- Remark 3: interest of shift-and-add with right shift for $\cos(x)$ and $\log_2(1+x)$ implementation
  $\rightsquigarrow$ reduction of 12.8 % and 13.75 %, respectively

Impact on the accuracy of some functions

$f(x)$	$\mathscr{I}$	d	$\log_2(\epsilon)$ not optimized	$\log_2(\epsilon)$ optimized
$\exp(x)-1$	$[-0.25, 0.25]$	7	-26.98	-27.34
$\exp(x)$	$[0, 1]$	7	-13.94	-14.90
$\sin(x)$	$[-0.5, 0.5]$	9	-18.95	-19.91
$\cos(x)$	$[-0.5, 0.25]$	5	-27.01	-27.26
$\tan(x)$	$[0.25, 0.5]$	9	-18.81	-19.64
$\log_2(1+x)/x$	$[2^{-23}, 1]$	7	-13.94	-14.89
$\sqrt{1+x}$	$[2^{-23}, 1]$	7	-13.94	-14.90

Table: Impact of the accuracy based selection step on the certified accuracy of the generated code for various functions.

- Remark 1: with a `mulacc` that computes `(a * b) + (c >> n)` with $n \in \{1, \cdots, 31\}$ with one final rounding
- Remark 2: more accurate results for all the cases, almost up to 1 bit of accuracy for *exp*, *sin*, $\log_2$ and $\sqrt{1+x}$

Outline of the talk

A strategy to synthesize code for matrix inversion

Let M be a matrix of fixed-point variables,

to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute B a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
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The basic blocks we need to include in our tool-chain

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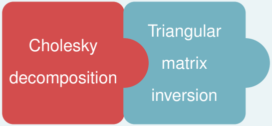
The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion



A strategy to synthesize code for matrix inversion

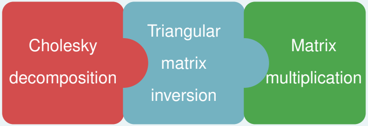
Let M be a matrix of fixed-point variables,

to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

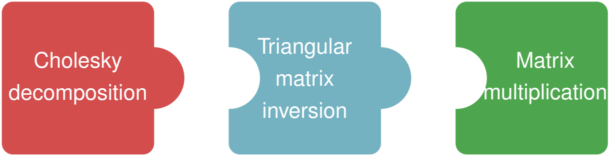
1. Generate certified code to compute B a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$

The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication



Linear algebra basic blocks



Linear algebra basic blocks

Matrix multiplication

Inputs

- Two matrices A and B of fixed-point variables

$$A \in \mathbb{F}\mathrm{ix}^{n \times n} \quad \text{and} \quad B \in \mathbb{F}\mathrm{ix}^{n \times n}$$

- A bound $\mathscr{C}_1$ on the roundoff error
- A bound $\mathscr{C}_2$ on the code size

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Output

- Fixed-point code (C, VHDL, ...) that evaluates the product

$$C' = A' \cdot B', \quad \text{where} \quad A' \in A \quad \text{and} \quad B' \in B$$

  that satisfy both $\mathscr{C}_1$ and $\mathscr{C}_2$

- Accuracy certificate (verifiable by a formal proof checker)

Straightforward algorithms

Accurate algorithm

- Main idea: a dot product code for each coefficient of the resulting matrix

Accurate algorithm

Inputs:

Two matrices $A \in \mathbb{F}ix^{n \times n}$ and $B \in \mathbb{F}ix^{n \times n}$

Outputs:

C code to compute the product $A \cdot B$

$n \cdot n$ accuracy certificates

Steps:

1: **for** $1 < i \leq n$ **do**
2:     **for** $1 < j \leq n$ **do**
3:         $DPSynthesis(A_{i,:}, B_{:,j})$
4:     **end for**
5: **end for**
6: *Check* $\mathscr{C}_1$ *and* $\mathscr{C}_2$

Straightforward algorithms

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4:     **end for**
5: **end for**
6: *Check* $\mathscr{C}_1$ *and* $\mathscr{C}_2$

Compact algorithm

- Main idea: a unique dot product code for all the coefficient of the resulting matrix

Compact algorithm

Inputs:

Two matrices $A \in \mathbb{F}ix^{n \times n}$ and $B \in \mathbb{F}ix^{n \times n}$

Outputs:

C code to compute the product $A \cdot B$

1 accuracy certificate

Steps:

1: $\mathscr{U} = A_{1,:} \cup A_{2,:} \cup \cdots \cup A_{n,:}$, with $\mathscr{U} \in \mathbb{F}ix^{1 \times n}$
2: $\mathscr{V} = B_{:,1} \cup B_{:,2} \cup \cdots \cup B_{:,n}$, with $\mathscr{V} \in \mathbb{F}ix^{n \times 1}$
3: $\mathit{DPSynthesis}(\mathscr{U}, \mathscr{V})$
4: *Check* $\mathscr{C}_1$ *and* $\mathscr{C}_2$

Example of a multiplication of 2×2 matrices

We consider code synthesis for the multiplication of matrices $A' \in A$ and $B' \in B$:

$$A = \begin{pmatrix} [-1000, 1000] & [-3000, 3000] \\ [-1, 1] & [-1, 1] \end{pmatrix} \text{ and } B = \begin{pmatrix} [-2000, 2000] & [-2, 2] \\ [-4000, 4000] & [-10, 10] \end{pmatrix}$$

Coefficient	$A_{1,1}$	$A_{1,2}$	$A_{2,1}$	$A_{2,2}$	$B_{1,1}$	$B_{1,2}$	$B_{2,1}$	$B_{2,2}$
Fixed-point format	$\mathbf{Q}_{11.21}$	$\mathbf{Q}_{12.20}$	$\mathbf{Q}_{2.30}$	$\mathbf{Q}_{2.30}$	$\mathbf{Q}_{11.21}$	$\mathbf{Q}_{3.29}$	$\mathbf{Q}_{2.30}$	$\mathbf{Q}_{5.27}$

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Accurate algorithm

Dot-product	$A_{1,:} \cdot B_{:,1}$	$A_{1,:} \cdot B_{:,2}$	$A_{2,:} \cdot B_{:,1}$	$A_{2,:} \cdot B_{:,2}$
Evaluated using	$\text{DPCode}_{1,1}$	$\text{DPCode}_{1,2}$	$\text{DPCode}_{2,1}$	$\text{DPCode}_{2,2}$
Output format	$Q_{26,6}$	$Q_{18,14}$	$Q_{15,17}$	$Q_{7,25}$
Certified error	$\approx 2^{-5}$	$\approx 2^{-14}$	$\approx 2^{-16}$	$\approx 2^{-24}$
Maximum error	$\approx 2^{-5}$			
Average error	$\approx 2^{-7}$			

Compact algorithm

Dot-product	$A_{1,:} \cdot B_{:,1}$	$A_{1,:} \cdot B_{:,2}$	$A_{2,:} \cdot B_{:,1}$	$A_{2,:} \cdot B_{:,2}$
Evaluated using	$\text{DPCode}_{\mathscr{U},\mathscr{V}}$			
Output format	$Q_{26,6}$			
Certified error	$\approx 2^{-5}$			
Maximum error	$\approx 2^{-5}$			
Average error	$\approx 2^{-5}$			

Trade-off algorithms

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \qquad B = \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} \\ b_{10} & b_{11} & b_{12} & b_{13} & b_{14} \\ b_{20} & b_{21} & b_{22} & b_{23} & b_{24} \\ b_{30} & b_{31} & b_{32} & b_{33} & b_{34} \\ b_{40} & b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

$A_{0:}$ $A_{1:}$ $A_{4:}$ $A_{2:}$ $A_{3:}$

Accurate algorithm:

(25 dot-product codes)

$B_{:0}$ $B_{:1}$ $B_{:4}$ $B_{:2}$ $B_{:3}$

Trade-off algorithms

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$$B = \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} \\ b_{10} & b_{11} & b_{12} & b_{13} & b_{14} \\ b_{20} & b_{21} & b_{22} & b_{23} & b_{24} \\ b_{30} & b_{31} & b_{32} & b_{33} & b_{34} \\ b_{40} & b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

$A_{0:}$ $A_{1:}$ $A_{4:}$ $A_{2:}$ $A_{3:}$

Compact algorithm:

(1 dot-product code)

$B_{:0}$ $B_{:1}$ $B_{:4}$ $B_{:2}$ $B_{:3}$

Trade-off algorithms

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(9 dot-product codes)

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Number of possible trade-off algorithms

- Given by $\mathscr{B}(n)^2$ with $\mathscr{B}(n)$ the n^{th} Bell number

n	5	6	10	16	25	64	···
$\mathscr{B}(\mathbf{n})^2$	2704	41 209	$\approx 2^{34}$	$\approx 2^{66}$	$\approx 2^{124}$	$\approx 2^{433}$	···

Trade-off algorithms

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

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How to find in reasonable time a partition that reduces cost size without harming the accuracy?

Distances between fixed-point variables

The Hausdorff distance

$$d_H : \mathbb{F}ix \times \mathbb{F}ix \rightarrow \mathbb{R}^+$$
$$d_H(I_1, I_2) = max\left\{\left|\underline{I_1} - \underline{I_2}\right|, \left|\overline{I_1} - \overline{I_2}\right|\right\}$$

Fixed-point distance

$$d_F : \mathbb{F}ix \times \mathbb{F}ix \rightarrow \mathbb{N}$$
$$d_F(I_1, I_2) = \left|IntegerPart(I_1) - IntegerPart(I_2)\right|$$

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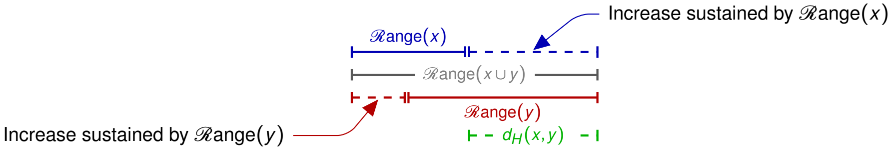

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Closest pair algorithm

Input:

Two matrices $A \in \mathbb{F}\mathrm{ix}^{n \times n}$ and $B \in \mathbb{F}\mathrm{ix}^{n \times n}$
An accuracy bound $\mathscr{C}_1$ (ex. the average error bound is $< \epsilon$)
A code size bound $\mathscr{C}_2$
A metric d

Output:

Code to compute $A \cdot B$ s.t. $\mathscr{C}_1$ and $\mathscr{C}_2$ are satisfied,
or no code otherwise

Algorithm:

$\mathscr{S}_A \leftarrow \{A_{0,:}, \ldots, A_{n-1,:}\}$
$\mathscr{S}_B \leftarrow \{B_{:,0}, \ldots, B_{:,n-1}\}$
while $\mathscr{C}_1$ is satisfied **do**
    $(u_A, v_A), d_A \leftarrow \mathit{findClosestPair}(\mathscr{S}_A, d)$
    $(u_B, v_B), d_B \leftarrow \mathit{findClosestPair}(\mathscr{S}_B, d)$
    if $d_A \leq d_B$ **then**
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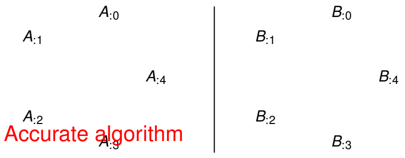
    end if
    for $(A_i, B_j) \in \mathscr{S}_A \times \mathscr{S}_B$ **do**
        $\mathit{DPSynthesis}(A_i, B_j)$
    end for
end while
/* Revert the last merging step, and check the bound $\mathscr{C}_2$. */



25 DPcodes

$\mathscr{C}_1$ is satisfied

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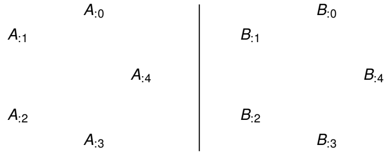

20 DPcodes

$\mathscr{C}_1$ is satisfied

Closest pair algorithm

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Two matrices $A \in \mathbb{F}\mathrm{ix}^{n\times n}$ and $B \in \mathbb{F}\mathrm{ix}^{n\times n}$
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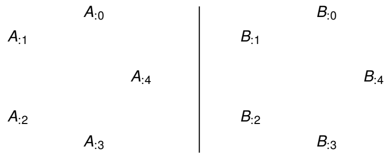

16 DPcodes

$\mathscr{C}_1$ is satisfied

Closest pair algorithm

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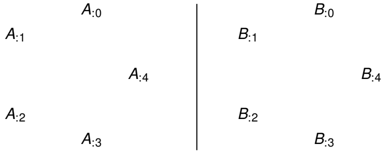

12 DPcodes

$\mathscr{C}_1$ is satisfied

Closest pair algorithm

Input:

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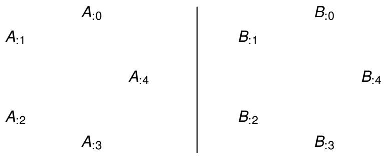

9 DPcodes

$\mathscr{C}_1$ is no longer satisfied

Closest pair algorithm

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  end if
  for $(A_i, B_j) \in \mathscr{S}_A \times \mathscr{S}_B$ **do**
    DPSynthesis(A_i, B_j)
  end for
end while
/* Revert the last merging step, and check the bound $\mathscr{C}_2$. */

$A_{:0}$ $A_{:1}$ $A_{:4}$ $A_{:2}$ $A_{:3}$ | $B_{:0}$ $B_{:1}$ $B_{:4}$ $B_{:2}$ $B_{:3}$

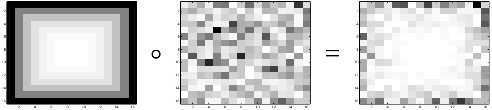
12 `DPcodes`

$\mathscr{C}_1$ is satisfied

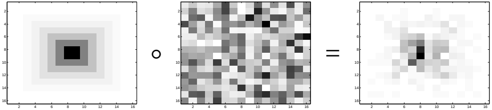
$\rightsquigarrow$ Revert the last merging step and check if $\mathscr{C}_2$ is satisfied

Benchmark matrices

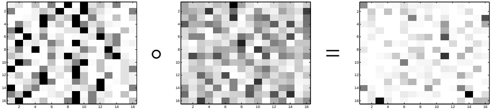
Edges



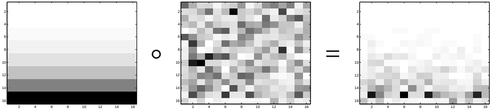
Center



Random

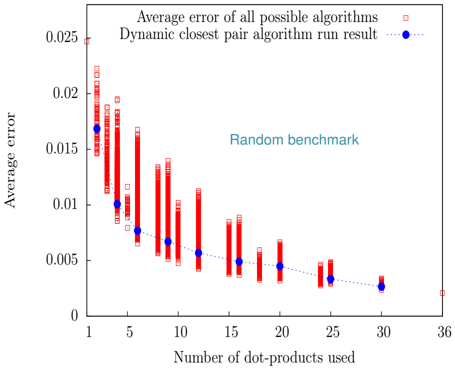


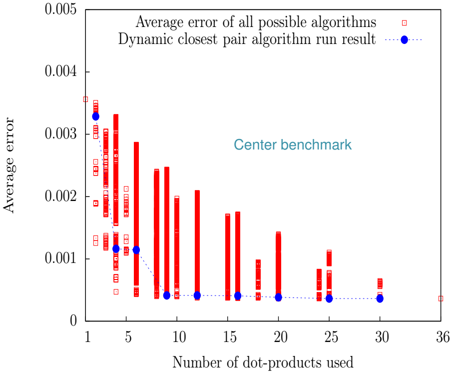
Rows



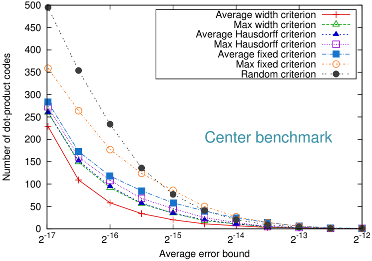
Efficiency of the distance-based heuristic: 6×6 matrix product

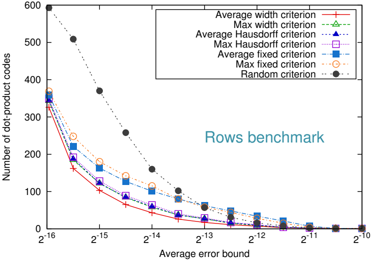
- Generating the 41 209 different algorithms: 2h15 per benchmark
- Running our algorithm: 10 seconds per benchmark

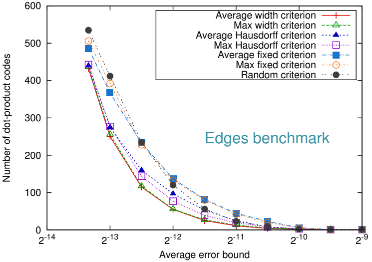


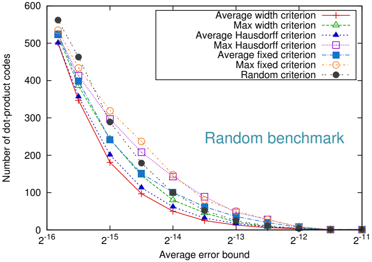


Impact of the metric on the trade-off strategy

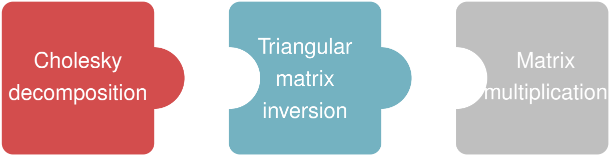








Linear algebra basic blocks



Cholesky decomposition and triangular matrix inversion

Cholesky decomposition

$$b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \dfrac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases}$$

$$\text{with } c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}$$

Triangular matrix inversion

$$n_{i,j} = \begin{cases} \dfrac{1}{b_{i,i}} & \text{if } i = j \\ \dfrac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases}$$

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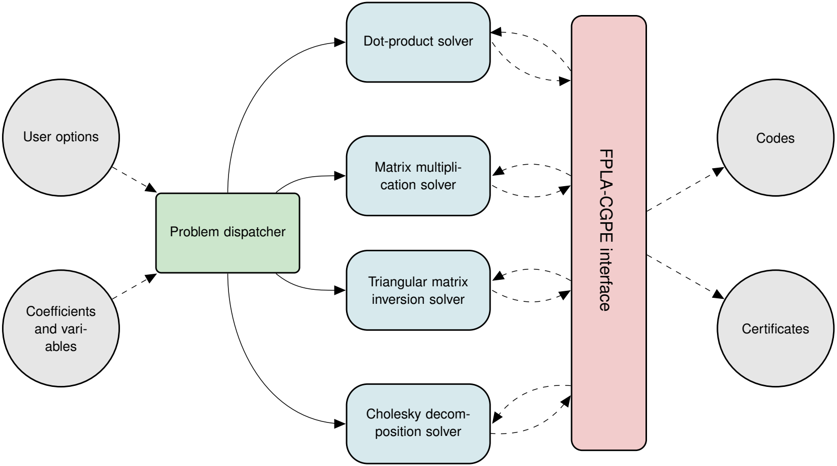
Dependencies of the coefficient $b_{4,2}$ in the decomposition and inversion of a 6×6 matrix.

FPLA (Fixed-Point Linear Algebra)

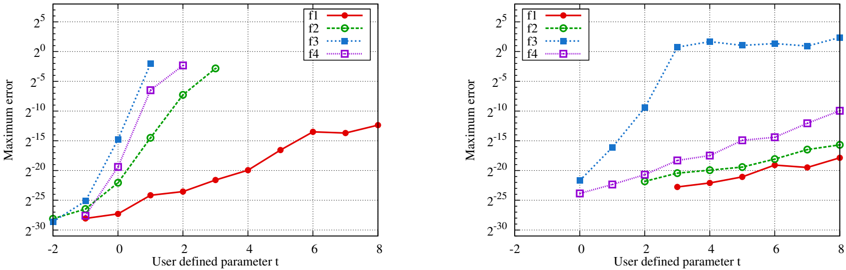


Impact of the output format of division

Different functions to set the output format of division

1. $f_1(i_1, i_2) = t$,
2. $f_2(i_1, i_2) = \min(i_1, i_2) + t$,
3. $f_3(i_1, i_2) = \max(i_1, i_2) + t$,
4. $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + t$,

i_1 and i_2: integer parts of the numerator and denominator and $t \in [-2, 8]$

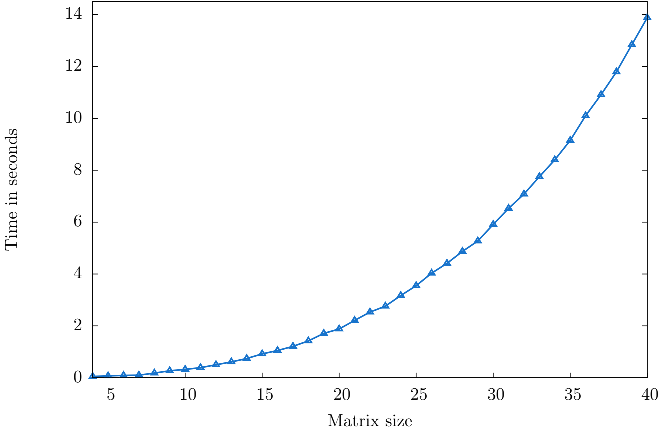


(a) Cholesky 5×5.   (b) Triangular 10×10.

Maximum errors with various functions used to determine the output formats of division.

How fast is generating triangular matrix inversion codes?

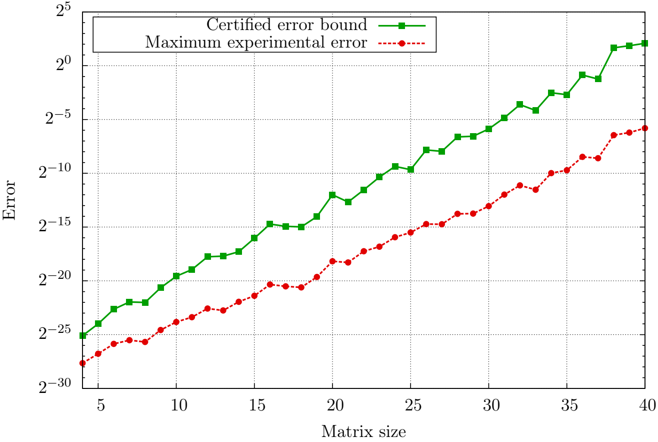
- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division



Generation time for the inversion of triangular matrices of size 4 to 40.

How fast is generating triangular matrix inversion codes?

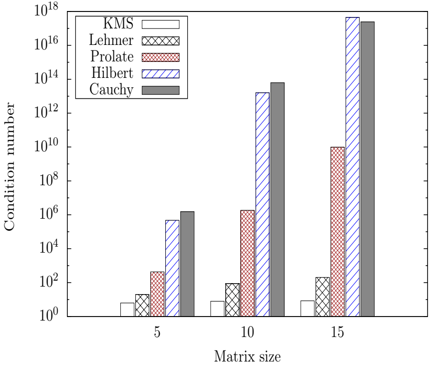
- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division



Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.

Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer



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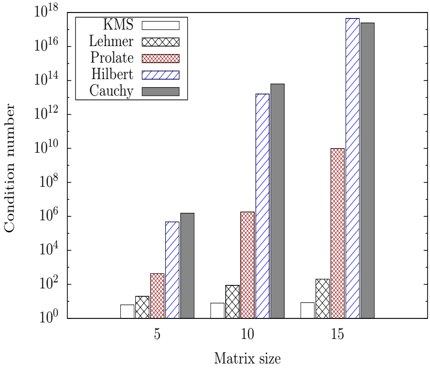

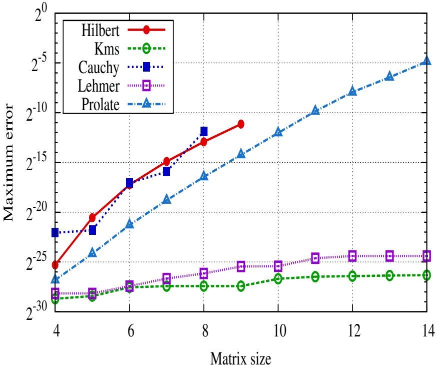

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate

Outline of the talk

Contributions

A framework for certified fixed-point code synthesis

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{}$ and $/$

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    - → matrix multiplication: accuracy vs. code size trade-offs,
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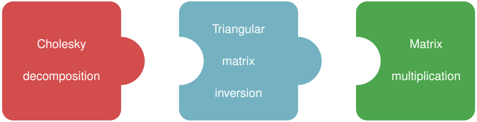
Publications

- DASIP14: *Toward the synthesis of fixed-point code for matrix inversion based on Cholesky decomposition* [?]
- SYNASC14: *Automated Synthesis of Target-Dependent Programs for Polynomial Evaluation in Fixed-Point Arithmetic* [?]
- PECCS14: *Code Size and Accuracy-Aware Synthesis of Fixed-Point Programs for Matrix Multiplication* [?]
- DASIP12: *Design of Fixed-point Embedded Systems (DEFIS) French ANR Project.* [?]
- SCAN12: *Approach based on instruction selection for fast and certified code generation* [?]

Perspectives

- Integrate the matrix inversion flow



Perspectives

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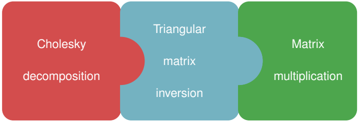
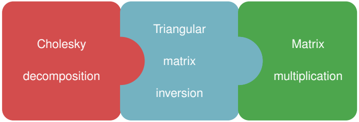


  - Extend the code size vs. accuracy trade-off strategy to Cholesky decomposition and triangular matrix inversion

Perspectives

- Integrate the matrix inversion flow



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- Extend the arithmetic model to support complex arithmetic
  - Objective: inverting co-variance matrices for Space Time Adaptive Processing

Perspectives

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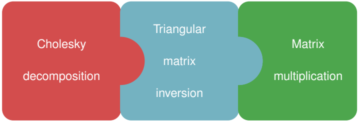

  - Extend the code size vs. accuracy trade-off strategy to Cholesky decomposition and triangular matrix inversion

- Extend the arithmetic model to support complex arithmetic
  - Objective: inverting co-variance matrices for Space Time Adaptive Processing

- Target hardware implementations
  - Suggest an arithmetic model for fully custom word-length variables
  - Code size vs. accuracy → chip area vs. accuracy

MERCI

[Wil98] H. Keding, M. Willems, M. Coors, and H. Meyr.
Fridge: a fixed-point design and simulation environment.

[IEEE754] IEEE 754.
IEEE Standard for Floating-Point Arithmetic.

[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys.
Automatic floating-point to fixed-point conversion for DSP code generation.

[IBMK10] Ali Irturk, Bridget Benson, Shahnam Mirzaei, and Ryan Kastner.
GUSTO: An Automatic Generation and Optimization Tool for Matrix Inversion Architectures.

[LHD12] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier.
Sum-of-products evaluation schemes with fixed-point arithmetic, and their application to IIR filter implementation.

[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen.
Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.

[MRS12] Daniel Ménard, Romuald Rocher, Olivier Sentieys, Nicolas Simon, Laurent-Stéphane Didier, Thibault Hilaire, Benoît Lopez, Eric Goubault, Sylvie Putot, Franck Vedrine, Amine Najahi, Guillaume Revy, Laurent Fangain, Christian Samoyeau, Fabrice Lemonnier, and Christophe Clienti.
Design of Fixed-Point Embedded Systems (defis) French ANR Project.

[LHD14] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier.
Formatting bits to better implement signal processing algorithms.

[Rev09] Guillaume Revy.
Implementation of binary floating-point arithmetic on embedded integer processors - Polynomial evaluation-based algorithms and certified code generation.

[MNR12] Christophe Mouilleron, Amine Najahi, and Guillaume Revy.
Approach based on instruction selection for fast and certified code generation.

[MR11] Christophe Mouilleron and Guillaume Revy.
Automatic Generation of Fast and Certified Code for Polynomial Evaluation.

[KG08] David R. Koes and Seth C. Goldstein.
Near-optimal instruction selection on DAGs.

[MNR14b] Matthieu Martel, Amine Najahi, and Guillaume Revy.
Toward the synthesis of fixed-point code for matrix inversion based on cholesky decomposition.

[MNR14c] Christophe Mouilleron, Amine Najahi, and Guillaume Revy.
Automated Synthesis of Target-Dependent Programs for Polynomial Evaluation in Fixed-Point Arithmetic.

[MNR14a] Matthieu Martel, Amine Najahi, and Guillaume Revy.
Code Size and Accuracy-Aware Synthesis of Fixed-Point Programs for Matrix Multiplication.

[CG09] Jason Cong, Karthik Gururaj, Bin Liu 0006, Chunyue Liu, Zhiru Zhang, Sheng Zhou, and Yi Zou.
Evaluation of static analysis techniques for fixed-point precision optimization.

[LV09] Dong-U Lee and John D. Villasenor.
Optimized custom precision function evaluation for embedded processors.